Lecture 05
12.5 Lines and Planes in space

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January 25, 2019

Things to note

MW 2:40-3:40
T 9:30-10:30
R 12:30-1:30
F 8:30-9:30
Also any time after class or by appointment

'Direction in textbook' comment

$\text{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}})$

Last Class

Cross Products

$$\vec{\mathbf{u}}\times\vec{\mathbf{v}} = (\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\sin(\theta))\vec{\mathbf{n}} = (u_2v_3-u_3v_2)\vec{\mathbf{i}}-(u_1v_3-u_3v_1)\vec{\mathbf{j}}+(u_1v_2-u_2v_1)\vec{\mathbf{k}}$$

$$= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Lines

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}_0+t\vec{\mathbf{v}} = \langle x_0, y_0, z_0\rangle+t\langle v_1, v_2, v_3\rangle = \langle x_0+tv_1, y_0+tv_2, z_0+tv_3\rangle.$$

Line example

Example

Find a vector equation for the line through $(4, -3, 5)$ *parallel to the line* $\vec{\mathbf{r}}(t) = \langle 2 + t, 8 - 2t, 3 \rangle$.

Line example

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Find a vector equation for the line through $(4, -3, 5)$ *parallel to the line* $\vec{\mathbf{r}}(t) = \langle 2 + t, 8 - 2t, 3 \rangle$.

$\vec{\mathbf{r}}(t) = \langle 4, -3, 5 \rangle + t\langle 1, -2, 0 \rangle = \langle 4 + t, -3 - 2t, 5 \rangle$.

Line example

Example

Find parametric equations for the line L through $(3, 5, 6)$ perpendicular to both $\vec{\mathbf{r}}_1(t) = \langle 1 - t, 2 - 3t, 5 - t \rangle$ and $\vec{\mathbf{r}}_2(t) = \langle -1 - 2t, 2, 7 + 2t \rangle$.

Line example

Example

Find parametric equations for the line L through $(3, 5, 6)$ *perpendicular to both* $\vec{\mathbf{r}}_1(t) = \langle 1 - t, 2 - 3t, 5 - t \rangle$ *and* $\vec{\mathbf{r}}_2(t) = \langle -1 - 2t, 2, 7 + 2t \rangle$*.*

The direction vector is the cross product of the direction vectors of $\vec{\mathbf{r}}_1(t)$ and $\vec{\mathbf{r}}_2(t)$.

Line example

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$$\vec{\mathbf{v}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ -1 & -3 & -1 \\ -2 & 0 & 2 \end{vmatrix} = \vec{\mathbf{i}}(-6-0) - \vec{\mathbf{j}}(-2-2) + \vec{\mathbf{k}}(0-6) = \langle -6, 4, -6 \rangle$$

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Thus the parametric equations for L are
$x = 3 - 6t, y = 5 + 4t, z = 6 - 6t$.

Planes

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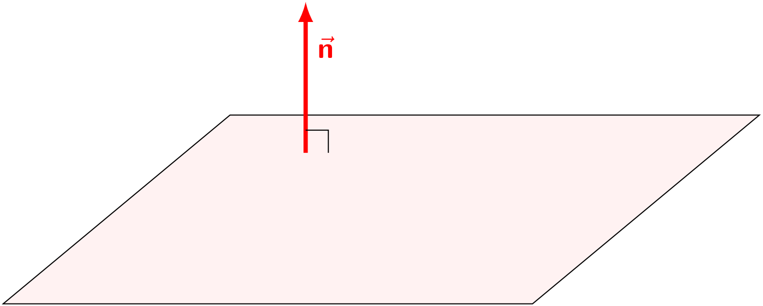
Definition

A vector is normal to a plane if it is orthogonal (perpendicular) to the plane.



Planes

We will take $P_0 = (x_0, y_0, z_0)$ to be a fixed point in the plane and $P_1 = (x, y, z)$ to be any other point in the plane.

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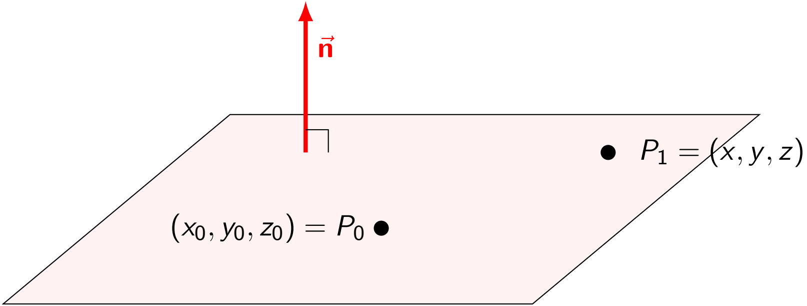

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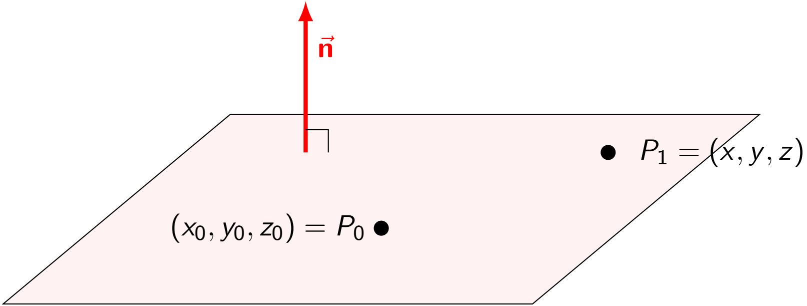


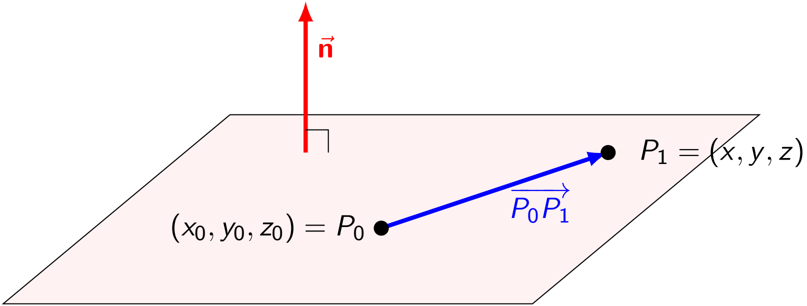
Question

What does it mean for P_1 to be a point in the plane? What vector is in the plane?

Planes

Answer

$\overrightarrow{P_0P_1}$ *is a vector in the plane.*



Planes

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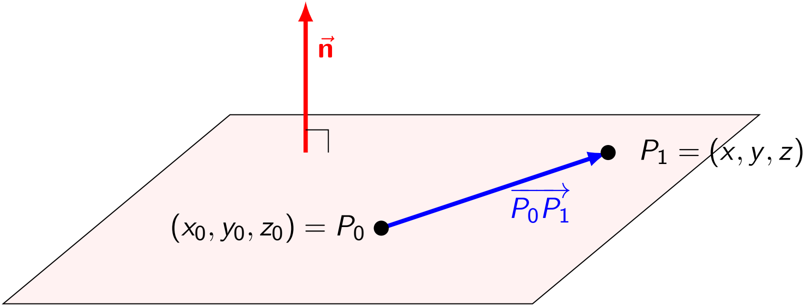


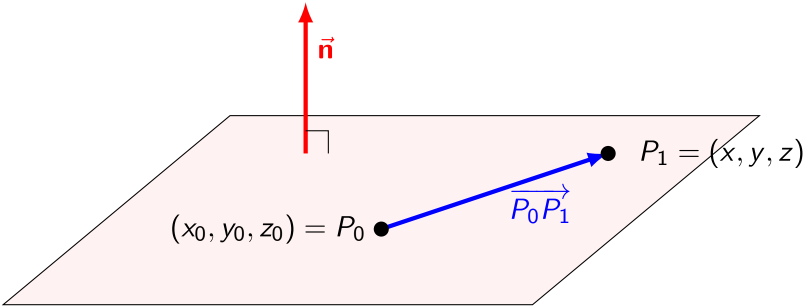
Question

What relationship do $\vec{\mathbf{n}}$ *and* $\overrightarrow{P_0P_1}$ *have?*

Planes

Answer

$\vec{\mathbf{n}}$ and $\overrightarrow{P_0P_1}$ are orthogonal.



Planes

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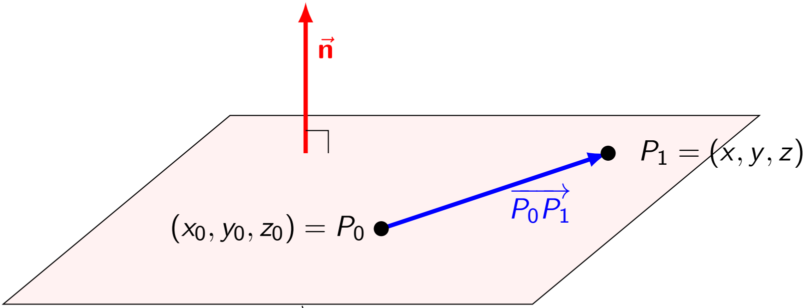

This means that $\vec{\mathbf{n}} \cdot \overrightarrow{P_0P_1} = 0$.

Equation of a plane

Definition

Let $\vec{\mathbf{n}} = \langle A, B, C \rangle$ *be a normal vector to a plane containing the point* $P_0 = (x_0, y_0, z_0)$*. Then the equation of the plane (where* $P_1 = (x, y, z)$*) is*

$$\vec{\mathbf{n}} \cdot \overrightarrow{P_0P_1} = 0$$

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or

$\langle A, B, C \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0,$ *the vector equation of a plane*

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$\langle A, B, C\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0,$ *the vector equation of a plane*

or

$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ *the coordinate form of a plane*

or

$Ax + By + Cz = Ax_0 + By_0 + Cz_0$ *the coordinate form simplified.*

Plane example

Example

Find the equation of the plane through $(-3, 0, 7)$ *with normal vector* $\vec{\mathbf{n}} = 5\vec{\mathbf{i}} + 2\vec{\mathbf{j}} - \vec{\mathbf{k}}$.

Plane example

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Find the equation of the plane through $(-3, 0, 7)$ *with normal vector* $\vec{\mathbf{n}} = 5\vec{\mathbf{i}} + 2\vec{\mathbf{j}} - \vec{\mathbf{k}}$.

The vector equation is $\langle 5, 2, -1\rangle \cdot \langle x + 3, y - 0, z - 7\rangle = 0$. We can simplify this to its coordinate form $5(x + 3) + 2(y) - (z - 7) = 0$, or $5x + 2y - z = -22$.

Plane example

Question

What is the normal vector of the xy-plane? What is the normal vector of the plane $3x - 6y = z + 8$*?*

Plane example

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What is the normal vector of the xy*-plane? What is the normal vector of the plane* $3x - 6y = z + 8$*?*

$\vec{\mathbf{k}}$ and $\langle 3, -6, -1 \rangle$ (or $\langle -3, 6, 1 \rangle$).

Plane example

Example

Find the equation of the plane through $R = (0, 0, 1)$*,* $S = (2, 0, 0)$*, and* $T = (0, 3, 0)$*.*

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Two vectors in the plane are $\overrightarrow{RS} = \langle 2, 0, -1 \rangle$ and $\overrightarrow{RT} = \langle 0, 3, -1 \rangle$.

Plane example

Example

Find the equation of the plane through $R = (0, 0, 1)$*,* $S = (2, 0, 0)$*, and* $T = (0, 3, 0)$*.*

Two vectors in the plane are $\overrightarrow{RS} = \langle 2, 0, -1 \rangle$ and $\overrightarrow{RT} = \langle 0, 3, -1 \rangle$.

$$\overrightarrow{RS} \times \overrightarrow{RT} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 3 & -1 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} \vec{\mathbf{k}} = \langle 3, 2, 6 \rangle.$$

Plane example

Example

Find the equation of the plane through $R = (0, 0, 1)$, $S = (2, 0, 0)$, and $T = (0, 3, 0)$.

Two vectors in the plane are $\overrightarrow{RS} = \langle 2, 0, -1 \rangle$ and $\overrightarrow{RT} = \langle 0, 3, -1 \rangle$.

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Thus the equation of the plane is

$$\langle 3, 2, 6, \rangle \cdot \langle x - 0, y - 0, z - 1 \rangle = 0, \text{ or } 3x + 2y + 6z = 6.$$

Notice you could use any of the given points.